5: Lemke-Howson

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Lemke-Howson: An algorithm for finding a NE in bimatrix games.

Step 1: Show reduction to symm. games
Step 2: Write down LP for which "special" vertices correspond to NE in the symm. game
Step 3: Find a "special" vtx. in the polytope given by the LP.

Symmetric Games:

Defn: A bimatrix game R, C is called **symmetric** if $R = C$

Thus for a symmetric game, row player's payoff for the mixed strategy (x, y) = col player's payoff for the strategy (y, x). NOT THE SAME AS IDENTICAL GAMES!
Also, both $x, y \in \Delta_n$

Given a bimatrix game (R, C), $R \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times m}$, $R, C > 0$, Consider the symmetric game

$$\begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} 0 & C \\ R & 0 \end{bmatrix} \end{array} = R' = C'.$$

Lemma: Let $((w^T, x^T), (z^T, y^T))$ be a NE for R', where $w, z \in \mathbb{R}^n$, $x, y \in \mathbb{R}^m$

(i) If $z \neq 0$, then $\left(\frac{x}{\|x\|_1}, \frac{z}{\|z\|_1}\right)$ is a NE for R, C

(ii) Else, $\left(\frac{y}{\|y\|_1}, \frac{w}{\|w\|_1}\right)$ is a NE for R, C

Proof: Assume $z \neq 0$.

Consider $R'\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} 0 & C \\ R & 0 \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} = \begin{array}{c} 1 \\ m \end{array}\begin{bmatrix} Cy \\ Rz \end{bmatrix}$

Since (w^T, x^T) is BR, $x_i > 0 \Rightarrow (Rz)_i = \max_{k \in [m]} (Rz)_k$

Hence for (R, C), x is BR to z

Also, $R'\begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} Cx \\ Rw \end{bmatrix}$

And since (z^T, y^T) is BR, $z_j > 0 \Rightarrow (Cx)_j = \max_{k \in [n]} (Cx)_k$

Hence for (R, C), z is BR to x.

Thus if $z \neq 0$, then (x, z) is a NE for (R, C)

(note: if $x = 0$, then $w \neq 0$, and hence $Cx = 0$, $Rw \neq 0$, & $\exists i : (Rw)_i > 0$. Then $z \neq 0$ cannot be a BR)

Similarly, if $z = 0$, then $y \neq 0$, & following above, $w \neq 0$. And we can show $\left(\frac{y}{\|y\|_1}, \frac{w}{\|w\|_1}\right)$ is a NE for R, C. ∎

Hence, to find a NE for (R, C), suffices to find eq. in symmetric games.

Let (R, R), $R \in \mathbb{R}^{m\times m}$, be a symmetric game. Consider the polytope:

$$P: \quad \forall i \in [m], \quad x_i \geq 0$$
$$\quad\;\; \forall i \in [m], \quad (Rx)_i \leq 1$$

(As before, assume $R > 0$. Further, assume P is nondegenerate).

$\forall x \in P$, we say coordinate i is **represented** if either $x_i = 0$ or $(Rx)_i = 1$.
And $x \in P$ is a **democracy** if every coordinate is represented at x.
Note that 0^m is a democracy.
Finally, let $Q = \{x \in P$: coordinates $1 \ldots m-1$ are represented
coordinate m is NOT represented
some coordinate is represented twice $\}$

Q. Show that if $x \in Q$, then x must be a vertex of P.

Claim: If $v \neq 0$ is a democracy, then $\left(\frac{v}{\|v\|_1}, \frac{v}{\|v\|_1}\right)$ is a NE of (R, R)

Proof: Consider Rv, and some $i \in [m]$ s.t. $v_i > 0$. Then since v is a democracy, Then $(Rv) \leq 1$ & $(Rv)_i = 1$, hence v is supported on max-payoff strategies & hence is a BR.

Lemke - Howson Algorithm:

1. Start at vertex $v_0 = (0, 0, \ldots, 0)$ w/ m tight constraints (since P is nondegenerate).
2. "Untighten" the constraint $x_m = 0$. The $(m-1)$ tight constraints that define an edge of the polytope, one end of which is $v = 0$. Find the other end, say v' (How?)
3. At the next vtx. v', the following holds:
   (i) Either v' is a democracy, or every coordinate $1 \ldots m-1$ is represented, and exactly one coordinate k is represented twice, i.e., $x_k = 0$ & $(Rx)_k = 1$
   (ii) Exactly one of these two equalities was not tight (i.e., was a strict inequality at the previous vertex. All the other tight inequalities were tight at the previous vtx.
   (iii) Coordinate m is not represented
   Hence, $v' \in Q$
4. Untighten the inequality for coordinate k that was tight at the earlier vertex, and proceed along the resulting edge for the next vertex.
5. Stop when you reach a democracy.

Let $v^0, v^1, \ldots, v^{t-1}, v^t, \ldots$ be the sequence of vertices visited.

Claim: Except the first & last vertex, all vertices $v^1, v^2, \ldots$ are in Q.

Proof: Since each v^i is a vertex, exactly m tight constraints at each.
Claim is true of v^1. Assume true for v^{t-1}, will show it is true for v^t.
- By algo, for coordinates $1 \ldots m-1$, a tight constraint continues to be tight at v^t.
- Since v^t is a vertex, P is nondegenerate, exactly one more constraint will be tight.
  If this is a $x_m \geq 0$ or $(Rx)_m \leq 1$, then v^t is a democracy & algo ends.
  Else, for $k \in [m-1]$, one additional constraint is tight. Hence $v^t \in Q$.

Lastly, we need to show that the algo doesn't cycle.

Lemma: The algorithm stops at a democracy that is not 0^m.

Proof: Consider the graph $G = (V, E)$ where:
- each vertex $v \in P$ is a vertex $v \in V$ if either:
  a. v is a democracy, or
  b. $v \in Q$
- each democratic $v \in V$ has an edge to $w \in V$, where w is the vertex reached by untightening $x_m = 0$ or $(Rx)_m = 1$
- for each $v \in V$ that corresponds to vertex $v \in Q$, add 2 edges: if coordinate k is represented twice at v, then add edges to the two vertices obtained by untightening the 2 constraints for coordinate k.

Claim: Each vertex has degree 1 or 2 in the graph
(prove yourself)

Hence, the graph consists of paths & cycles.

Further, v has degree 1 in the graph iff v is a democracy.
Hence by the L-H algorithm, we will reach the other end of the path that starts at 0^m, which must be another democracy. ∎

Untightening a tight constraint:

Let $v \in Q$ be our current vertex. Let
$S_1 \subseteq [m]$ be s.t. $\forall i \in S_1$, $v_i = 0$
$S_2 \subseteq [m]$ be s.t. $\forall i \in S_2$, $(Rv)_i = 1$
& $k = S_1 \cap S_2$ (note that $S_1 \cup S_2 = [m-1]$).

Case I: Untightening $v_k = 0$
Let x be a soln. to the LP:

$$\forall i \in S_1 \setminus k \quad x_i = 0$$
$$\forall i \in S_2 \quad (Rx)_i = 0$$
$$x_k = 1$$

Claim: The LP is feasible
(prove yourself)

Then consider $v^\lambda = v + \lambda x$, for $\lambda \geq 0$

Then $\forall i \in S_1 \setminus k$, $v^\lambda_i = 0$
$\forall i \in S_2$, $(Rv^\lambda)_i = 1$
& $v^\lambda_k = \lambda$

Thus, $m-1$ constraints $1 \ldots m-1$ are tight $\forall \lambda \geq 0$.
Increase λ until a new constraint becomes tight. This is the new vertex. Can check this is in Q.

Case II: Untightening $(Rv)_k = 1$
Similar to the previous case, consider the LP:

$$\forall i \in S_1 \quad x_i = 0$$
$$\forall i \in S_2 \setminus k \quad (Rx)_i = 0$$
$$(Rx)_k = -1$$

Increase λ until for $v + \lambda x$ a new constraint becomes tight.
This is the new vertex. Can check that this is also in Q.